Prime Maltsev conditions

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Example

The variety $\mathcal{BA}$ is the class of all Boolean algebras $(B; \wedge, \vee, ', 0, 1)$ defined by the usual associativity, commutativity, distributivity, absorption, identity and complements equations.

Example

The variety $\mathcal{BR}$ is the class of all Boolean rings $(R; +, \cdot, 0, 1)$ defined by the usual associativity, commutativity, distributivity, identity and idempotency equations.

The two varieties are not the same, as they have a different number of operational symbols and thus different algebras, but they are equivalent. Every Boolean algebra can be turned into a boolean ring and vice versa:

$$x + y = (x \wedge y') \vee (x' \wedge y), \quad x \cdot y = x \wedge y, \quad 1 = 1, \quad 0 = 0.$$

Example

The variety $\mathcal{SET}$ is the class of all sets with no basic operations.

Example

The variety $\mathcal{SG}$ is the class of all semigroups $(S; \cdot)$ defined by associativity.

Example

The variety $\mathcal{SLAT}$ is the class of all semilattices $(S; \wedge)$ defined by associativity, commutativity and idempotency.

$\mathcal{SET}$ and $\mathcal{SG}$ are not term equivalent, but every semigroup can be turned into a set, and every set can be turned into a semigroup by defining $x \cdot y = x$.

Semigroups cannot be turned into semilattices, because semilattices satisfy the identity $x \wedge y = y \wedge x$ and such operation cannot be defined as a semigroup term.

Definition (W.D. Neumann; 1974)

Let Γ be a set of identities over a signature. We say that Γ **interprets in a variety** $\mathcal{K}$ if by replacing the operation symbols in Γ by some term expressions of $\mathcal{K}$, the so obtained set of identities holds in $\mathcal{K}$.

Definition

A **variety** $\mathcal{K}_1$ **interprets in a variety** $\mathcal{K}_2$, denoted as $\mathcal{K}_1 \preceq \mathcal{K}_2$, if there is a set of identities Γ that defines $\mathcal{K}_1$ and interprets in $\mathcal{K}_2$.

- The varieties $\mathcal{BA}$ and $\mathcal{BR}$ are equi-interpretable.
- The varieties $\mathcal{SET}$ and $\mathcal{SG}$ are equi-interpretable.
- The variety of groups interprets in the variety of Abelian groups.
- The variety $\mathcal{SET}$ interprets in any other variety.
- Every variety interprets in the variety of trivial algebras ($x \approx y$).
- Constants c are modelled by unary operations satisfying $c(x) \approx c(y)$.
- The interpretability relation $\preceq$ is a quasi-order on the class of varieties.

Lattice of interpretability types

Theorem

The class of varieties modulo equi-interpretability forms a bounded lattice, **the lattice of interpretability types**, *with* $\overline{\mathcal{V}} \vee \overline{\mathcal{W}} = \overline{\mathcal{V} \amalg \mathcal{W}}$ *and* $\overline{\mathcal{V}} \wedge \overline{\mathcal{W}} = \overline{\mathcal{V} \otimes \mathcal{W}}$.

Definition

The **coproduct** of the varieties $\mathcal{V} = \mathrm{Mod}\,\Sigma$ and $\mathcal{W} = \mathrm{Mod}\,\Delta$ in disjoint signatures is the variety $\mathcal{V} \amalg \mathcal{W} = \mathrm{Mod}(\Sigma \cup \Delta)$.

Definition

The **varietal product** of $\mathcal{V}$ and $\mathcal{W}$ is the variety $\mathcal{V} \otimes \mathcal{W}$ of algebras $\mathbf{A} \otimes \mathbf{B}$ for $\mathbf{A} \in \mathcal{V}$ and $\mathbf{B} \in \mathcal{W}$ whose

- universe is $A \times B$,
- basic operations are $s \otimes t$ acting coordinate-wise for each pair of n-ary terms of $\mathcal{V}$ and $\mathcal{W}$.

Theorem (O. Garcia, W. Taylor; 1984)

In the lattice of interpretability types of varieties

- *minimal element: sets (equi-interpretable with semigroups)*
- *maximal element: trivial algebras*
- *the class of idempotent varieties form a sublattice*
- *the class of finitely presented varieties forms a sublattice*
- *the class of varieties defined by linear equations forms a join sub-semilattice*
- *not modular*
- *meet prime elements: boolean algebras, lattices, semilattices*
- *meet irreducible elements: groups*
- *join prime elements: commutative groupoids, trivial algebras*

J. Mycielski (1977): Lattice of interpretability types of first order theories

Maltsev conditions

Definition

A **strong Maltsev condition** is a finite set Γ of equations. A variety $\mathcal{V}$ satisfies Γ iff Γ interprets in $\mathcal{V}$. Thus, strong Maltsev conditions are principal filters generated by finitely based varieties in the lattice of interpretability types.

Examples: Maltsev term, having a majority term, semilattice term, Siggers term, Olšák term, etc.

Definition

A **Maltsev condition** is a descending chain $\Gamma_0 \succeq \Gamma_1 \succeq \Gamma_2 \succeq \dots$ of strong Maltsev conditions. A variety $\mathcal{V}$ satisfies it if $\Gamma_i \preceq \mathcal{V}$ for some $n \in \mathbb{N}$. Thus, Maltsev conditions are filters in the lattice of interpretability types.

Examples: having a near-unanimity term, Taylor term, Jónsson terms, Gumm terms, edge term, Hagemann-Mitschke terms, etc.

Theorem (A.I. Maltsev; 1954)

For a variety $\mathcal{V}$ the following are equivalent:

- *$\mathcal{V}$ is congruence permutable ($\alpha \circ \beta = \beta \circ \alpha$ for all $\alpha, \beta \in \operatorname{Con} \mathbf{A}$),*
- *ϱ is symmetric for any $\mathbf{A} \in \mathcal{V}$ and reflexive relation $\varrho \leq \mathbf{A}^2$,*
- *$\mathcal{V}$ has a Maltsev term satisfying $x \approx p(x, y, y)$ and $p(x, x, y) \approx y$.*

Theorem (J. Hagemann, A. Mitschke; 1973)

For a variety $\mathcal{V}$ and $n \geq 2$ the following are equivalent:

- *$\mathcal{V}$ is congruence n-permutable ($\alpha \circ^n \beta = \beta \circ^n \alpha$ for $\alpha, \beta \in \operatorname{Con} \mathbf{A}$),*
- *$\varrho^{-1} \subseteq \varrho \circ^{n-1} \varrho$ for any $\mathbf{A} \in \mathcal{V}$ and reflexive relation $\varrho \leq \mathbf{A}^2$,*
- *$\mathcal{V}$ has ternary terms $p_1, \ldots, p_{n-1}$ satisfying*

$$\begin{aligned} x &\approx p_1(x, y, y), \\ p_i(x, x, y) &\approx p_{i+1}(x, y, y) \text{ for } 1 \leq i < n-1, \\ p_{n-1}(x, x, y) &\approx y. \end{aligned}$$

- O. Garcia and W. Taylor **conjectured** in 1984 that 2-permutability is prime in the lattice of interpretability types of varieties.
- S. Tschantz announced in 1996 a proof of the conjecture. However, his proof has remained unpublished.
- K. Kearnes and S. Tschantz: 2-permutability is prime in the lattice of interpretability types of **idempotent** varieties.
- M. Valeriote and R. Willard: n-permutability for some n is prime filter in the lattice of interpretability types of **idempotent** varieties.
- J. Opršal: for any $n \geq 2$, n-permutability is prime in the lattice of interpretability types of **linear** varieties.

Theorem

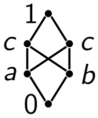
For $n \geq 5$, n-permutability is **not prime** *in the lattice of interpretability types of varieties.*

Theorem

2*-permutability* **is prime** *in the lattice of interpretability types of varieties.*

n-permutability is not prime for $n \geq 5$

- Let $\mathcal{O}$ be the variety generated by the order primal algebra of the following poset.



- Let $\mathcal{M}$ be the variety defined by the majority identities $m(y,x,x) \approx m(x,y,x) \approx m(x,x,y) \approx x$.
- Let $\mathcal{P}_n$ be the variety defined by the Hagemann-Mitschke terms.
- $\mathcal{P}_n \not\preceq \mathcal{O}$, because $\mathcal{O}$ has the above 6-element algebra and a compatible reflexive and transitive relation that is not symmetric.
- $\mathcal{P}_n \not\preceq \mathcal{M}$, by the same argument for the 2-element lattice.
- $\mathcal{O}$ has a 5-ary near-unanimity operation, it is a locally finite, congruence distributive, and quasiorder distributive variety. We show that every quasiorder in $\mathcal{O} \amalg \mathcal{M}$ is symmetric.
- $\mathcal{P}_5 \preceq \mathcal{O} \amalg \mathcal{M}$.

2-permutability is prime

Lemma

For any non-2-permutable variety $\mathcal{O}$, $\mathcal{O} \amalg \mathcal{M}$ is also non-2-permutable.

- It suffices to find an algebra $(A; \mathcal{F}) \in \mathcal{O}$ and a majority function $m : A^3 \to A$ such that $(A; \mathcal{F} \cup \{m\})$ still has no Maltsev term.
- We start from a reflexive non-symmetric compatible digraph $\mathbb{G}_0$ in $\mathcal{O}$.
- By a primitive-positive construction we define a digraph $\mathbb{G}_1$ whose connected components have universal source and sink vertices.
- We construct $\mathbb{G}_2$ whose components have a universal vertex connected to all other vertices with symmetric edges.
- $\mathbb{G}_2$ has a non-complete component (non-2-permutable), and using the universal vertices we can define a compatible majority operation.

Problem

Is there a non-3-permutable variety $\mathcal{O}$ such that $\mathcal{O} \amalg \mathcal{M}$ is 3-permutable?
We proved that such variety cannot be locally finite.

2-permutability is prime

- The **complement** of a digraph $\mathbb{G}$ is the digraph whose vertex set is G and whose edge set is $G^2 \setminus E(\mathbb{G})$.
- The **product** of the digraphs $\mathbb{G}$ and $\mathbb{H}$ is defined on $G \times H$ with $(g_1, h_1) \to (g_2, h_2) \iff g_1 \to g_2$ and $h_1 \to h_2$.
- We define the **power** $\mathbb{G}^{\mathbb{H}}$ of two digraphs as follows. The vertex set of $\mathbb{G}^{\mathbb{H}}$ is the set of maps from the set H to the set G. The edge relation of $\mathbb{G}^{\mathbb{H}}$ is defined by

$$f \to f' \text{ if and only if } f(x) \to f'(y) \text{ in } \mathbb{G} \text{ for every } x \to y \text{ in } \mathbb{H}.$$

Lemma

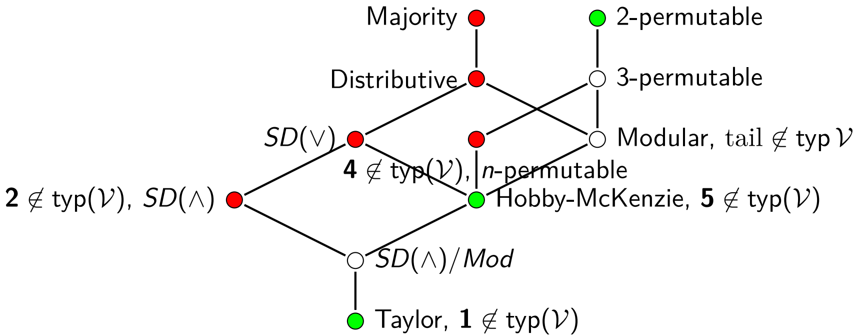
Let $\mathbb{G}$ and $\mathbb{H}$ be two digraphs with universal vertices u_G and u_H, respectively. Let $\mathbb{G}^*$ and $\mathbb{H}^*$ be the complements of the digraphs $\mathbb{G} - u_G$ and $\mathbb{H} - u_H$. Let κ be an infinite cardinal where $\kappa \geq \max(|G|, |H|)$, and let $\mathbb{K}$ be a complete digraph of κ-many vertices. Then

$$\mathbb{G}^{\mathbb{H}^* \times \mathbb{K}} \cong \mathbb{H}^{\mathbb{G}^* \times \mathbb{K}}.$$

2-permutability is prime

- It suffices to prove that the join of any two non-permutable varieties $\mathcal{G}$ and $\mathcal{H}$ is non-permutable.
- It suffices to construct a digraph that is compatible in both varieties and admits no Maltsev operation.
- By the first lemma, there are compatible digraphs $\mathbb{G}$ in $\mathcal{G}$ and $\mathbb{H}$ in $\mathcal{H}$ such that both of $\mathbb{G}$ and $\mathbb{H}$ have a non-complete component and each component of $\mathbb{G}$ and $\mathbb{H}$ has a universal vertex.
- Assume that $\mathbb{G}$ and $\mathbb{H}$ are connected (otherwise more constructions).
- By the second lemma, $\mathbb{G}^{\mathbb{H}^* \times \mathbb{K}} \cong \mathbb{H}^{\mathbb{G}^* \times \mathbb{K}}$. So we have a digraph that is a compatible both in $\mathcal{G}$ and $\mathcal{H}$ (note that this is not reflexive).
- The constant maps in $\mathbb{G}^{\mathbb{H}^* \times \mathbb{K}}$ induce a subgraph isomorphic to $\mathbb{G}$.
- If u is a universal vertex for a component in $\mathbb{G}$, $a \to u \to b$, and p is a Maltsev operation, then $a = p(a, u, u) \to p(u, u, b) = b$, which contradicts our assumption that $\mathbb{G}$ has a non-complete component.

Maltsev filters of varieties



- prime Maltsev filters:
  - congruence permutable, $m(x, y, y) \approx m(y, y, x) \approx x$
  - Hobby-McKenzie term (join semi-distributive over modular)
  - Taylor term, non-trivial idempotent Maltsev condition
- non-prime Maltsev filters:
  - congruence n-permutable for some n, Hagemann-Mitschke terms
  - congruence distributive $=$ join semi-distributive and modular
  - congruence join semi-distributive (K. Kearnes and E. W. Kiss)

Majority is not prime

Theorem

Congruence meet semi-distributivity, congruence join semi-distributivity, congruence distributivity and having a majority term are not prime in the lattice of interpretability types of varieties.

- Let $\mathcal{V}$ be the variety defined by the minority identities
$$m(x,y,y) \approx m(y,x,y) \approx m(y,y,x) \approx x.$$
- Let $\mathcal{W}$ be the variety defined by identities
$$s(x,x) \approx x, \qquad s(x,y) \approx s(y,x).$$
- We have $\mathbf{A} = (\mathbb{Z}_2; x+y+z) \in \mathcal{V}$ and $\mathbf{B} = (\mathbb{Z}_3; 2x+2y) \in \mathcal{W}$. $\mathrm{Con}\,\mathbf{A}^2 \cong \mathbf{M}_3$ and $\mathrm{Con}\,\mathbf{B}^2 \cong \mathbf{M}_4$, so $\mathcal{V}$ and $\mathcal{W}$ are not congruence meet semi-distributive.
- However, their join has a majority term:
$$m(s(x,y), s(y,z), s(z,x)).$$

Taylor is prime

Theorem (W. Taylor, 1977; J. Olšák, 2017)

For any variety $\mathcal{V}$ the following are equivalent

- $\mathcal{V}_{\mathrm{id}} \not\leq \mathcal{SET}$,
- *satisfies a non-trivial idempotent Maltsev condition,*
- *has a Taylor-term:* $t(x,\dots,x) \approx x$ *and* $t(\dots,x,\dots) \approx t(\dots,y,\dots)$,
- *has an Olšák term* $t(x,x,x,x,x,x) \approx x$ *and* $t(x,y,y,y,x,x) \approx t(y,x,y,x,y,x) \approx t(y,y,x,x,x,y)$.

Theorem

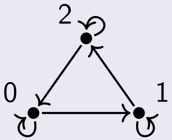
The filter of Taylor varieties is prime in the lattice of interpretability types.

Approach: Given two non-Taylor varieties $\mathcal{V}$ and $\mathcal{W}$, find a compatible digraph $\mathbb{G}$ in both $\mathcal{V}$ and $\mathcal{W}$ that does not admit a Taylor polymorphism.

Taylor is prime

Definition

Let $\mathbb{C} = (\{0, 1, 2\}; \to)$ be the reflexive directed 3-cycle.



Proposition

There are six essential polymorphisms of $\mathbb{C}$, the constants and the automorphisms. Thus, $\mathbf{C} = (C; \mathrm{Pol}(\mathbb{C}))$ generates a non-Taylor variety.

Proposition

If $\mathbb{F}$ is a compatible digraph of a variety $\mathcal{V}$ and $\mathbb{C}$ is a retract of $\mathbb{F}$, then $\mathcal{V}$ is non-Taylor.

Lemma

A variety is non-Taylor iff it has a compatible reflexive digraph $\mathbb{F}$ that has $\mathbb{C}$ as a retract.

- Let $\mathcal{V}$ be a non-Taylor variety
- $\mathbf{F}$ the free algebra in $\mathcal{V}$ freely generated by $\{x, y, z\}$
- ϱ the subalgebra of $\mathbf{F}^2$ generated by $\{xx, yy, zz, xy, yz, zx\}$
- $\mathbb{F} = (F; \varrho)$ is reflexive digraph
- $u \to v$ in $\mathbb{F}$ iff there exists a 6-ary term t in $\mathcal{V}$ so that

$$u(x, y, z) \approx t(x, y, z, x, y, z), \quad t(x, y, z, y, z, x) \approx v(x, y, z)$$

- In particular, if $u \to v$, then $u(x, x, x) \approx v(x, x, x)$
- $\mathbb{F}$ has as many (strong) components as there are unary terms in $\mathcal{V}$
- $\mathcal{V}_{\text{id}} \preceq \mathcal{SET}$, so there is a graph homomorphism form the idempotent component of $\mathbb{F}$ to $\mathbb{C}$
- $\mathbb{C}$ is reflexive, so this can be extended to an $\mathbb{F} \to \mathbb{C}$ homomorphism
- so $\mathbb{C}$ is a graph retract of $\mathbb{F}$
- $\mathcal{V} \preceq \text{Pol}(\mathbb{F})$ and $\text{Pol}_{\text{id}}(\mathbb{F}) \preceq \mathcal{SET}$

Taylor is prime

Theorem

For any variety $\mathcal{V}$ the following are equivalent:

- *$\mathcal{V}$ is non-Taylor,*
- *there are sets K_t $(t \in T)$, not all empty, such that $\dot{\bigcup}_{t\in T}\mathbb{C}^{K_t}$ is a compatible digraph in $\mathcal{V}$,*
- *for any sufficiently large infinite cardinals κ and τ the digraph $\dot{\bigcup}_{\mu\leq\kappa}\tau\mathbb{C}^{\mu}$ is a compatible digraph in $\mathcal{V}$.*

Corollary

The filter of Taylor varieties is prime in the lattice of interpretability types.

Problem

Describe the interpretability lattice for the varieties generated by the disjoint union of $\mathbb{C}$-powers.

Maltsev conditions in non-Taylor varieties

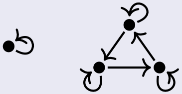
Proposition

- $\mathcal{SET} \lnsim \mathrm{Pol}(\mathbb{C})$ *because of the Maltsev condition* $u(x) \approx u(y)$.
- $\mathrm{Pol}(\mathbb{C}) \lnsim \mathrm{Pol}(\mathbb{C}^2)$ *because of the Maltsev condition*

$$f(f(x,y), f(y,z)) \approx y$$

  satisfied by the polymorphism $f(\overline{x_1x_2}, \overline{y_1y_2}) = \overline{x_2y_1}$ *of* $\mathbb{C}^2$.
- $\mathrm{Pol}(\mathbb{C}+1) \not\lesssim \mathrm{Pol}(\mathbb{C}^K)$ *because of the Maltsev condition*

$$e(x) \approx e(y), \quad t(e(x), y) \approx t(y, e(x)) \approx y.$$



- $\mathrm{Pol}(\mathbb{G}) \preceq \mathrm{Pol}(4\mathbb{C}+4)$, *for the reflexive 4-cycle digraph* $\mathbb{G}$.

Hobby-McKenzie is prime

Theorem (D. Hobby and R. McKenzie; 1988)

For any variety $\mathcal{V}$ the following are equivalent:

- *$\mathcal{V}$ has Hobby-McKenzie terms,*
- *$\mathcal{V}_{\mathrm{id}} \not\preceq \mathcal{SLAT}$.*

Theorem

Let $\mathbb{S}$ be a connected reflexive relational structure and $\mathcal{S}$ be the variety generated by $(S; \mathrm{Pol}(\mathbb{S}))$. If $\mathcal{V}_{\mathrm{id}} \preceq \mathcal{S}$, then $\mathcal{V}$ has a compatible relational structure $\mathbb{F}$ with $\mathbb{S}$ as a retract.

For the variety $\mathcal{SLAT}$ we have considered the following two structures:

$$\mathbb{D} = (\{0,1,2\}; \{00, 11, 22, 01, 10, 12, 20\}),$$
$$\mathbb{S} = (\{0,1\}; \{000, 010, 100, 111\}).$$

Hobby-McKenzie is prime

Theorem

For any variety $\mathcal{V}$ the following are equivalent

- *has no Hobby-McKenzie term.*
- *$\mathcal{V}$ has a compatible reflexive ternary hypergraph that has*

$$\mathbb{S} = (\{0,1\}; \{000, 010, 100, 111\})$$

  as a retract.
- *For any sufficiently large infinite cardinals κ and τ the digraph $\dot{\bigcup}_{\mu \leq \kappa} \tau \mathbb{S}^{\mu}$ is a compatible hypergraph in $\mathcal{V}$.*

Corollary

The filter of Hobby-McKenzie varieties is prime in the lattice of interpretability types.

Congruence and graph conditions

By a **congruence condition** we mean any property of varieties that can be expressed by the congruence relations of the algebras in the variety.

By a **digraph condition** we mean any property that can be expressed by the set of compatible directed graphs of the algebras in the variety.

- Congruence lattice operations vs the presence of individual digraphs.
- Maltsev conditions vs lack of “bad configurations”.

Problem

Which properties of varieties can be identified by the class of all compatible directed graphs in the variety?

Congruence permutable varieties

Proposition

For any variety $\mathcal{V}$ the following are equivalent:

- *V is congruence 2-permutable,*
- *every reflexive compatible digraph in $\mathcal{V}$ is symmetric (and transitive),*
- *in all compatible digraphs if $a \to b \leftarrow c \to d$, then $a \to d$.*

Definition

The **extreme congruence** of a digraph $\mathbb{G} = (G; \to)$ is $(\to \cap \leftarrow)^*$, the **strong congruence** is $\to^* \cap \leftarrow^*$, the **weak congruence** is $(\to \cup \leftarrow)^*$.

Proposition

A variety $\mathcal{V}$ is congruence n-permutable for some n iff the strong and weak congruences are the same in every reflexive compatible digraph in $\mathcal{V}$.

Taylor varieties

Theorem

A variety is Taylor iff all its reflexive antisymmetric digraphs are cycle free.

- ($\Leftarrow$): If $\mathcal{V}$ is not Taylor, then the free construction on $\mathbf{F}_{\mathcal{V}}(x, y, z)$ yields a reflexive digraph $\mathbb{F}$ that has $\mathbb{C}$ as a retract.
- We can find a factor $\mathbb{F}/\vartheta$ that is antisymmetric and contains $\mathbb{C}$.
- ($\Rightarrow$): Take a reflexive and antisymmetric digraph $\mathbb{G}$ that has a non-trivial directed cycle $\mathbb{C}$ of the smallest possible length.
- Can assume that $\mathbb{G}$ is generated by $\mathbb{C}$ with idempotent operations.
- Any finite subset $G_0 \subseteq G$ can be generated with a single idempotent operation f whose variables are all essential.
- Show that f must be the decomposition operation

$$f(f(x_{11}, \dots, x_{1n}), \dots, f(x_{n1}, \dots, x_{nn})) = f(x_{11}, x_{22}, \dots, x_{nn}).$$

- $f : \mathbb{C}^n \to \mathbb{G}$ must be injective, and $\mathbb{C}$ is a retract of $\mathbb{G}|_{G_0}$.
- Use compactness to show that $\mathbb{C}$ is a retract of $\mathbb{G}$.

Taylor varieties

Definition

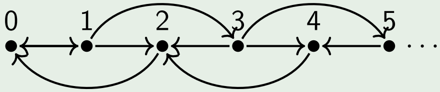
We can make any digraph $\mathbb{G}$ antisymmetric by repeatedly factoring by the extreme congruence. The $*$-**extreme congruence** of $\mathbb{G}$ is the smallest equivalence relation ϑ that makes $\mathbb{G}/\vartheta$ antisymmetric.

Theorem

A variety $\mathcal{V}$ is Taylor iff the $*$-extreme and strong congruences are the same in every compatible reflexive digraph in $\mathcal{V}$.

Example

The following digraph has a compatible semilattice with linear order, so need to factorize by the extreme congruence arbitrary many times.



Hobby-McKenzie varieties

Theorem

If $\mathcal{V}$ is congruence modular, then the strong and extreme congruences are the same in every reflexive compatible digraph.

Theorem

If the strong and extreme congruences are the same in every reflexive compatible digraph in $\mathcal{V}$, then $\mathcal{V}$ has Hobby-McKenzie terms.

Theorem

A locally finite variety $\mathcal{V}$ has Hobby-McKenzie terms if and only if the strong and extreme congruences are the same in every reflexive compatible digraph in $\mathcal{V}$.

Problem

Does the last theorem hold without the local finiteness assumption?

Further open problems

Problem

Are the following varieties prime in the lattice of interpretability types of varieties:

- congruence modularity,
- congruence 3-permutable, congruence 4-permutable,
- meet semi-distributive over modular?

Theorem (J. Opršal; 2017)

Congruence modularity is prime in the lattice of interpretability types of **idempotent** *varieties.*

Problem

Find a digraph condition (or relational structure condition) that characterizes congruence modularity, congruence distributivity, or having a majority term.

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